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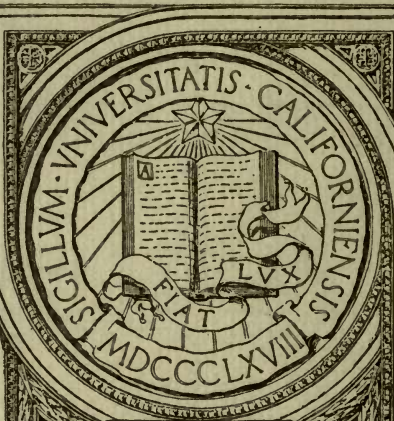
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THE MEASUREMENT OF THE COEFFICIENT OF VISCOSITY BY MEANS OF THE FORCED VIBRATIONS OF A SPHERE

BY

GEORGE FRANCIS MCEWEN

A.B., Stanford University, 1908



A THESIS

Submitted in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy in Physics at

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GEORGE FRANCIS MCGEE
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THE MEASUREMENT OF THE FRICTIONAL FORCE EXERTED ON A SPHERE BY A VISCOUS FLUID, WHEN THE CENTER OF THE SPHERE PERFORMS SMALL PERIODIC OSCILLATIONS ALONG A STRAIGHT LINE.

BY GEORGE F. McEWEN.

THE purpose of the following investigation was to devise a method by which Stokes's law of the frictional resistance of a fluid to the motion of a sphere whose center performs small periodic oscillations could be applied to the measurement of the coefficient of viscosity of fluids. The two main objects were: first, to obtain as close an agreement as possible between the actual working conditions and those demanded by theory; second, to devise a process of measuring the force acting on the sphere, even for fluids having a very large coefficient of viscosity.

The contents of this paper fall under the following five heads:

I. The effect of the internal friction of fluids on the motion of pendulums, from Sir G. G. Stokes's Math. and Phys. Papers, Cambridge, 1880 and 1901, Vols. I. and III.

II. An account of the method adopted in the present investigation to overcome the difficulties mentioned by Stokes, and to more nearly realize in the experimental work the ideal condition assumed in the theory from which Stokes's law was deduced.

III. Experimental tests of the present method.

IV. Suggestions for future research.

V. Summary of the paper.

I. THE EFFECT OF THE INTERNAL FRICTION OF FLUIDS ON THE MOTION OF PENDULUMS.

I. *Observations on the Motion of Pendulums.*

An account of the experiments made by Bessel, Baily, Dubuat, and Sabine, and the theoretical results obtained by Poisson, Challis and Plana is given in Stokes's Math. and Phys. Papers, Vol. III., pp. 1-7.

The effect of the surrounding fluid on the time of vibration of a pendulum was computed by Poisson, Challis, Green, and Plana from the hydrodynamical theory of a frictionless fluid. A fair agreement with the observations was found in some cases, but in many cases, especially

where the dimensions were small, the theory failed entirely to account for the experimental results.

Because of this failure, Stokes was led to apply the equations of motion¹ of a viscous fluid to pendulum problems.

2. Stokes's Deduction of the Law of Resistance.²

In 1850 Stokes completed the solution of the following problem:

The center of a sphere performs small periodic oscillations along a straight line; the sphere having a motion of translation only; it is required to determine from the ordinary hydrodynamic equations of motion of a viscous fluid, the motion of the surrounding fluid, and the force exerted on the sphere.

He assumed the velocities to be so small that their squares could be neglected, that there was no slipping of the fluid along the surface of the solid in contact with it, and that the amplitude of vibration of the sphere was very small and remained constant.

His result for the force is

$$F = -A \frac{d^2y}{dt^2} - B \frac{dy}{dt},$$

where y is the displacement and t is the time, and the coefficients A and B have the following values:

$$A = \left(\frac{1}{2} + \frac{9}{4\lambda r} \right) M_1' = K_1' M_1',$$

$$B = \left(\frac{9}{4\lambda r} \right) \left(1 + \frac{1}{\lambda r} \right) M_1' a = K_1'' M_1' a,$$

where M_1' = the mass of the fluid displaced by the sphere,

r = the radius of the sphere,

$a = \frac{2\pi}{T}$, where T is the period of oscillation, $\lambda = \sqrt{\frac{a}{2\mu'}}$, and

μ' = coefficient of viscosity $\div$ density.

From the same assumptions and equations, the same result has been obtained by a different method.³

For a cylinder oscillating in a direction perpendicular to its axis, Stokes deduced the following expression:

$$F = -K_2' m_2' \frac{d^2y}{dt^2} - K_2'' m_2' a \frac{dy}{dt},$$

where F = the force acting upon unit length of the cylinder, m_2' = the

¹ Stokes's Math. and Phys. Papers, Vol. I., pp. 75-105.

² Stokes's Math. and Phys. Papers, Vol. III., pp. 11-36.

³ Lamb's Hydrodynamics, edition 3, pp. 583-584.

mass of the fluid displaced by unit length of the cylinder, and K_2' and K_2'' are constants depending on the radius of the cylinder, the period of oscillation and μ' .

No simple expressions for K_2' and K_2'' were found that covered all cases, and they are best determined by special series and tables which are given by Stokes in his *Math. and Phys. Papers*, Vol. III., pp. 47-54. In the same volume he gives a discussion of the conditions upon which the theory from which the above expression was deduced depends.

In the case of the cylinder as well as the sphere, the first coefficient shows that the fluid has the same effect as an increase of the inertia of the system, and the second coefficient shows that the fluid opposes the motion of the system by a force proportional to the velocity.

From observations on the time of vibration the quantity A can be computed, and B can be computed from observations on the arc. Both A and B can be calculated from theory with the aid of Stokes's equations.¹

3. *The Application of Stokes's Law to the Pendulum Observations of Bessel, Baily and Dubuat.*²

By choosing a constant value of μ' Stokes calculated from his theory the periods of a variety of pendulums swinging in air, and the agreement with the values observed by Bessel and Baily was very satisfactory. Only a few observations on the arc of vibration were available, and these were only approximate, in these cases he calculated the decrement of the arc of vibration from his theory, and found as good an agreement with the experiments as could be expected considering the character of the observations. Though most of the observations made on pendulums oscillating in water were also in fair agreement with his theory, there were a number of discrepancies, which he attributed to the experimental methods used.

When a pendulum oscillates in water, or a more viscous liquid, the arc of oscillation rapidly decreases; this diminution forms, in fact, the greatest difficulty in experiments of this kind. This difficulty, which made it impossible to test his theory or to determine the coefficient of viscosity of water or of more viscous liquids by means of a vibrating body, suggested the present investigation.

II. THE METHOD ADOPTED IN THE PRESENT INVESTIGATION FOR MEASURING THE FORCE ACTING ON AN OSCILLATING SPHERE.

1. *Description of the Apparatus.*

A light rigid bar³ *FED* is secured to a knife-edge at *E*, which rests

¹ Stokes's *Math. and Phys. Papers*, Vol. III., pp. 1-141.

² Stokes's *Math. and Phys. Papers*, Vol. III., pp. 76-140.

³ See Fig. 1 for a diagram of the apparatus.

upon a fixed horizontal support. A second bar MN , perpendicular to the first one, carries two movable weights W_3 and W_4 ; the upper one can be moved to or from E by turning it about the screw EN , and a scale fastened to MN is provided for recording the distance of W_3 from E . The lower weight W_4 can be slid along EM and clamped in any desired

position. At F and D (points in line with E) two scale-pans W_1 and W_2 are attached by means of very thin strips of steel. A second light rigid bar ABC is attached by means of similar strips of steel at C and A , to the pan W_2 , and to a support above A , which can be given a vertical periodic motion of small amplitude. This support does not touch the apparatus, except at A . The sphere is suspended in the fluid by means of a fine wire, as shown; the wire being attached by a thin steel strip, to a small support B , which can be fastened to the bar AC , in any position between A and C , but is always in line with A and C .

By adding weights to the scale-pans, the system can be brought to a state of equilibrium when FD is horizontal. By adjusting the weights W_3 and W_4 , the natural period of oscillation

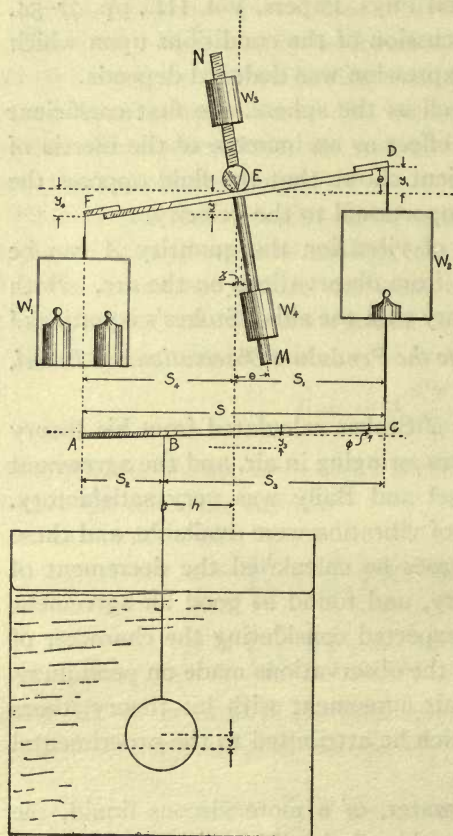


Fig. 1.

can be varied, as this adjustment changes the height of the center of gravity of the system rigidly attached to the knife-edge.

The oscillations will, from the construction of the apparatus, cause the sphere to move in a vertical line, and if the support of A is fixed, these oscillations will subside because of the friction. But if the support of A is given a vertical periodic motion, the system will be set in forced vibration, and the amplitude, phase and period of this forced vibration will depend upon the amplitude, phase and period of the motion of the support, for any given adjustment of the vibrating system.

2. The Theory of the Forced Vibrations of the Above System.

Explanation of the Symbols:

M'' equals the mass rigidly connected to the knife-edge E .

I_C'' equals the moment of inertia of M'' about C_1 .

I_E'' equals the moment of inertia of M'' about E .

C_1 is the center of mass of M'' .

EC_1 equals l .

m_3' equals the mass of the lower bar.

I_3' equals the moment of inertia of m_3' about its center of mass.

y_3 equals the displacement of the center of mass of m_3' from the position of equilibrium.

M_1 equals the mass of the sphere and its suspending wire.

θ equals the angular displacement of FD from its equilibrium position which is assumed to be horizontal.

M_1' equals the mass of the fluid displaced by the sphere.

g equals the acceleration of gravity

$$s = s_2 + s_3; \quad h = \frac{2}{s} - s_2.$$

θ_0 equals the angular displacement of the heavy pendulum used to maintain the vibrations.

$-l_1\theta_0$ equals y_0 , the linear displacement of the point A .

β equals the change in torque due to the elasticity of one suspending strip when bent through unit angle.

$\beta_1 y$ equals the change in the force on the wire holding the sphere, due to the elasticity of the surface film of the liquid, and to the varying length of the wire immersed.

The variable force exerted on the sphere by the fluid equals

$$-A \frac{d^2 y}{dt^2} - B \frac{dy}{dt}$$

Denote the variable force exerted on the wire by the fluid by:

$$-A' \frac{d^2 y}{dt^2} - B' \frac{dy}{dt} + \beta_1 y.$$

ϕ equals the angular displacement of the lower bar from the equilibrium position.

$-k_2 \frac{d\phi}{dt}$ = the torque on the lower bar due to the friction of the air and the suspending wires.

$-k_1 \frac{d\theta}{dt}$ = the torque on the upper bar due to the friction of the air, the suspending wires, and the knife-edge.

F_E' equals the horizontal force at E acting on the knife-edge.

F_E equals the vertical force at E acting on the knife-edge.

F_A equals the vertical force at A acting on the lower bar.

— F_B equals the vertical force at B acting on the lower bar.

F_C equals the vertical force at C acting on the lower bar.

— F_D equals the vertical force at D acting on the upper bar.

— F equals the vertical force at F acting on the upper ~~part~~ *bar*.

m_3 equals the mass of each scale-pan and contents, and wires between C and D .

m_4 equals the mass of each scale pan and contents, and wires attached at F .

The meaning of the other symbols used is indicated on the diagram of the apparatus.

Fundamental Dynamical Relations:

$$M_1 \frac{d^2 y}{dt^2} = F - M_1 g + M_1' g - (A + A') \frac{d^2 y}{dt^2} - (B + B') \frac{dy}{dt} + \beta_1 y,$$

$$m_3' \frac{d^2 y_3}{dt^2} = -F_B + F_A + F_C - m_3' g,$$

$$I_3' \frac{d^2 \phi}{dt^2} = F_C \frac{S}{2} - F_A \frac{S}{2} + F_B h - 3\beta \phi - k_2 \frac{d\phi}{dt},$$

$$m_3 \frac{d^2 y_1}{dt^2} = F_D - F_C,$$

$$I_C'' \frac{d^2 \theta}{dt^2} = -F_D S_1 + F S_4 - F_E' L \cos \theta - F_E L \sin \theta - k_1 \frac{d\theta}{dt} - 2\beta \theta \\ + FL \sin \theta + F_D l \sin \theta,$$

$$M'' \frac{d^2 y}{dt^2} = M'' \left\{ (\cos \theta) \left(\frac{d\theta}{dt} \right)^2 + (\sin \theta) \frac{d^2 \theta}{dt^2} \right\} l = F_E - F - F_D - M'' g,$$

$$M'' \frac{d^2 x}{dt^2} = M'' \left\{ (\cos \theta) \frac{d^2 \theta}{dt^2} - (\sin \theta) \left(\frac{d\theta}{dt} \right)^2 \right\} l = F_E',$$

$$m_2 \frac{d^2 y_4}{dt^2} = F - m_2 g,$$

$$I_C'' \frac{d^2 \theta}{dt^2} = S_4 \left(m_2 \frac{d^2 y_4}{dt^2} + m_2 g \right) - S_1 \left\{ F_C + m_3 \frac{d^2 y_1}{dt^2} \right\} - k_1 \frac{d\theta}{dt} - 2\beta \theta \\ - M'' l^2 (\cos \theta) \left\{ (\cos \theta) \frac{d^2 \theta}{dt^2} - (\sin \theta) \left(\frac{d\theta}{dt} \right)^2 \right\} \\ - l (\sin \theta) \left\{ M'' l \left[(\cos \theta) \left(\frac{d\theta}{dt} \right)^2 + (\sin \theta) \frac{d^2 \theta}{dt^2} \right] + M'' g \right\}.$$

Fundamental Kinematical Relations:

$\underline{x}$ and $\underline{y}$ are the coördinates of C_1 ,

$$\underline{x} = l \sin \theta, \quad \underline{y} = -l \cos \theta,$$

$$\frac{d\underline{x}}{dt} = l(\cos \theta) \frac{d\theta}{dt}, \quad \frac{d\underline{y}}{dt} = l(\sin \theta) \frac{d\theta}{dt},$$

$$\frac{d^2\underline{x}}{dt^2} = l \left\{ (\cos \theta) \frac{d^2\theta}{dt^2} - (\sin \theta) \left(\frac{d\theta}{dt} \right)^2 \right\},$$

$$\frac{d^2\underline{y}}{dt^2} = l \left\{ (\sin \theta) \frac{d^2\theta}{dt^2} + (\cos \theta) \left(\frac{d\theta}{dt} \right)^2 \right\},$$

$$y_1 = S_1\theta, \quad \frac{dy_1}{dt} = S_1 \frac{d\theta}{dt}, \quad \frac{d^2y_1}{dt^2} = S_1 \frac{d^2\theta}{dt^2},$$

$$y_4 = -S_4\theta, \quad \frac{dy_4}{dt} = -S_4 \frac{d\theta}{dt}, \quad \frac{d^2y_4}{dt^2} = -S_4 \frac{d^2\theta}{dt^2},$$

$$y = \frac{1}{S} (S_3y_0 + S_2y_1) = \frac{1}{S} (-S_3l_1\theta_0 + S_2S_1\theta),$$

$$y_3 = \frac{1}{2}(y_0 + y_1), \quad y_0 = -l_1\theta_0,$$

$$\phi = -\left(\frac{y_0 - y_1}{S} \right) = \frac{1}{S} (l_1\theta_0 + S_1\theta).$$

$$\frac{dy}{dt} = \frac{1}{S} \left(S_1S_2 \frac{d\theta}{dt} - S_3l_1 \frac{d\theta_0}{dt} \right),$$

$$\frac{d^2y}{dt^2} = \frac{1}{S} \left(S_1S_2 \frac{d^2\theta}{dt^2} - S_3l_1 \frac{d^2\theta_0}{dt^2} \right),$$

$$\frac{dy_3}{dt} = \frac{1}{2} \left(S_1 \frac{d\theta}{dt} - l_1 \frac{d\theta_0}{dt} \right),$$

$$\frac{d^2y_3}{dt^2} = \frac{1}{2} \left(S_1 \frac{d^2\theta}{dt^2} - l_1 \frac{d^2\theta_0}{dt^2} \right),$$

$$\frac{d\phi}{dt} = \frac{1}{S} \left(l_1 \frac{d\theta_0}{dt} + S_1 \frac{d\theta}{dt} \right),$$

$$\frac{d^2\phi}{dt^2} = \frac{1}{S} \left(l_1 \frac{d^2\theta_0}{dt^2} + S_1 \frac{d^2\theta}{dt^2} \right).$$

The above dynamical and kinematical relations can be so combined as to give the following ordinary differential equation:

$$P \frac{d^2\theta}{dt^2} + P_1 \frac{d\theta}{dt} + P_2\theta = P'' \frac{d^2\theta}{dt^2} + P' \frac{d\theta_0}{dt} + P_2\theta_0,$$

in which the coefficients are constants and have the following values:

$$\begin{aligned}
 P &= \left[I_E'' + m_2 S_4^2 + m_3 S_1^2 + \frac{S_1^2 S_2^2}{S^2} (M_1 + A + A') + \frac{S_1^2}{4} m_3' + I_3' \frac{S_1^2}{S^2} \right], \\
 P_1 &= \left[\frac{S_1^2 S_2^2}{S^2} B + \left\{ \frac{S_1^2 S_2^2}{S^2} B' + k_1 + k_2 \left(\frac{S_1}{S} \right)^2 \right\} \right], \\
 P_2 &= \left[M'' gl + \beta \left(2 + 3 \left[\frac{S_1}{S} \right]^2 \right) - \frac{S_1^2 S_2^2}{S^2} \beta_1 \right], \\
 P'' &= \left[\frac{l_1 S_1 S_2 S_3}{S^2} (M_1 + A + A') + \frac{l_1 S_1}{4} m_3' - \frac{l_1 S_1}{S^2} I_3' \right], \\
 P' &= \left[(B + B') \frac{l_1 S_1 S_2 S_3}{S^2} - \frac{l_1 S_1}{S^2} k_2 \right], \\
 \underline{P}_2 &= - \left[\frac{3 l_1 S_1 \beta}{S^2} + \frac{l_1 S_1 S_2 S_3}{S^2} \beta_1 \right].
 \end{aligned}$$

Assume

$$y_0 = ce^{-at} \sin at = -l_1 \theta_0,$$

and substitute in the second member of the differential equation; the result will be:

$$P \frac{d^2 \theta}{dt^2} + P_1 \frac{d\theta}{dt} + P_2 \theta = P_0'' \left(-\frac{c}{l_1} \right) e^{-at} \sin at + P_0' \left(-\frac{c}{l_1} \right) e^{-at} \cos at,$$

where

$$P_0'' = (\underline{P}_2 - [a^2 - \alpha^2] P'' - \alpha P'), \text{ and } P_0' = a(P' - 2\alpha P'').$$

Assume that the following ordinary equation is a particular integral of the differential equation:

$$\theta = A_1 e^{-at} \sin (at - \theta_1) + B_1 e^{-at} \cos (at - \theta_2),$$

substitute in the equation and solve for the quantities:

$$A_1, \theta_1, B_1, \text{ and } \theta_2.$$

The following values of the constants will make the assumed form for θ , a solution:

$$\begin{aligned}
 A_1 &= \frac{P_0'' (\sec \theta_1) \left(-\frac{c}{l_1} \right)}{P_2 + P[\alpha^2 - a^2] - \alpha P_1 + (P_1 a - 2a\alpha P) \tan \theta_1}, \\
 B_1 &= \frac{P_0' (\sec \theta_2) \left(-\frac{c}{l_1} \right)}{P_2 + P[\alpha^2 - a^2] - \alpha P_1 + (P_1 a - 2a\alpha P) \tan \theta_2}, \\
 \tan \theta_1 &= \frac{P_1 a - 2a\alpha P}{P_2 + P[\alpha^2 - a^2] - \alpha P_1} = \tan \theta_2
 \end{aligned}$$

To obtain the complete integral of the equation, the complementary function, $A_3 e^{-P_1 t/2P} \sin(\beta_3 t + \theta_3)$ would have to be added to the above value of θ . A_3 and θ_3 are arbitrary constants, depending for their values on how the system is started, and

$$\beta_3 = \frac{\sqrt{4P_2P - P_1^2}}{2P}.$$

In the present method of measurement, no observations are taken until the complementary function has a negligible value, so only the first or particular integral will be retained.

In practical applications, the ratio of the maximum value of θ to the maximum value of θ_0 is determined by experiment. The relation of the coefficient B to that ratio, which is required for determining B will now be derived. The following equation for θ_0 is assumed:

$$\theta_0 = -\frac{c}{l_1} e^{-at} \sin at.$$

Then the expression for θ can be reduced to the form

$$\theta = A_2 e^{-at} \sin(at - \theta_1 + \theta_3)$$

where

$$A_2 = \sqrt{A_1^2 + B_1^2}, \quad \text{and} \quad \tan \theta_3 = \frac{B_1}{A_1} = \frac{P_0'}{P_0''}.$$

Let t_1 be the time at which θ has a maximum value, and t_2 be the time at which θ_0 first reaches its maximum value after t_1 . Let $t_2 - t_1 = \Delta t$.

By taking the first derivative of θ and θ_0 , the maximum values, θ' and θ_0' given below are obtained:

$$\theta' = A_2 e^{-at_1} \left(\frac{1}{\sqrt{1 + \left(\frac{\alpha}{a}\right)^2}} \right), \quad \theta_0' = -\frac{c}{l_1} e^{-at_2} \left(\frac{1}{\sqrt{1 + \left(\frac{\alpha}{a}\right)^2}} \right).$$

Dividing the first equation by the second gives

$$\frac{\theta'}{\theta_0'} = R' = e^{a\Delta t} \left(\frac{A_2}{-\frac{c}{l_1}} \right).$$

This ratio is obtained experimentally by first observing the maximum value of θ , then observing the next maximum value of θ_0 , and dividing the first by the second. Substituting the value of A and B in the expression for A gives

$$A_2 = -\left(\frac{c}{l_1}\right) \frac{\sqrt{(P_0')^2 + (P_0'')^2}}{\sqrt{(P_2 + P[\alpha^2 - a^2] + \alpha P_1)^2 + (P_1 a - 2a\alpha P)^2}},$$

and

$$R' = e^{a\Delta t} \frac{\sqrt{(P_0')^2 + (P_0'')^2}}{\sqrt{(P_2 + P[\alpha^2 - a^2] - \alpha P_1)^2 + (P_1 a - 2a\alpha P^2)}}.$$

This expression shows that R' is independent of the amplitude of the vibration, but is a function of the constant coefficients of one of the preceding differential equations.

3. *The Deduction of a Practical Working Formula from the Above Theory.*

It appears that P_2 is the constant by which θ is multiplied to obtain that part of the torque which opposes the displacement of the system, but does not depend on the friction. In the expression

$$P_2 = M''gl + \beta \left(2 + 3 \left[\frac{S_1}{S} \right]^2 \right) - \frac{S_1^2 S_3}{S^2} \beta_1$$

l is the distance above E of the center of mass of the system rigidly attached to the knife-edge. As will be seen by referring to the expression for R' , for the values of the constants in the expression for R' , the quantity l appears only in the expression for P_2 . Therefore, by shifting the weight W_3 , l and consequently $M''gl$ can be changed without affecting the remainder of the expression for R' .¹

Let

$$R = e^{-a\Delta t} R' = \frac{\sqrt{(P_0')^2 + (P_0'')^2}}{\sqrt{(P_2 + P[\alpha^2 - a^2] - \alpha P_1)^2 + (P_1 a - 2a\alpha P^2)}},$$

now if R is measured for three different values of $M''gl$, there will be three equations from which P_1 can be computed by eliminating the other unknown quantities.

Suppose $M''gl$ is changed by raising or lowering the weight W_3 , a known amount; the amount due to one complete turn, for example. Denote this change by $\sqrt{2_2}$. Then

$$M''l = (M'' - W_3)l + W_3l_3,$$

where l is the distance of the center of the mass of M'' from E , l is the distance of the center of mass of $(M'' - W_3)$ from E , and l_3 is the distance of the center of mass of W_3 from E . Let the change in l_3 be Δl_3 , and denote the corresponding change in l by Δl , then l will remain the same as it depends only on that part of the mass whose position with reference

¹ Changing l in this way would produce a known small change in P , which can be easily compensated for by changing the weights in the scale-pans W_1 and W_2 . It will be assumed that this is done, and therefore P will be treated as a constant.

to E is fixed. Therefore,

$$M''(l + \Delta l) = (M'' - W_3)l + W_3(l_3 + \Delta l_3),$$

$$\Delta l = \frac{W_3}{M''} \Delta l_3,$$

and

$$\sqrt{z_2} = M'' g \Delta l = W_3 g \Delta l_3.$$

In the equation for R , denote the numerator by $\sqrt{z_3}$, denote $(aP_1 - 2a\alpha P)^2$ by z_1 , denote $(P_2 + P[\alpha^2 - a^2] - \alpha P_1)$ by $n\sqrt{z_2} - z'$,¹ and let $+\frac{z'}{\sqrt{z_2}} = w$. Then

$$(P_2 + P[\alpha^2 - a^2] - \alpha P_1) = \sqrt{z_2}(n - w),$$

and the equation for R takes the following simplified form:

$$R_n^2 = \frac{z_3}{z_2(n - w)^2 + z_1}.$$

Now by varying n , the number of complete turns of W_3 , R_n will vary but z_2 , z_3 , and z_1 will remain constant, and are positive. w is a constant, positive or negative, and depends on the position of W_3 when n is assumed to be zero.

The maximum value of R , denoted by R_w is:

$$R_w = \sqrt{\frac{z_3}{z_1}},$$

therefore R_w is a function of z_1 and z_3 only, and corresponds to the special case when there is resonance. The above equation involving R_n can be written in the following forms:

$$(n - w)^2 + V \left[1 - \left(\frac{R_w}{R_n} \right)^2 \right] = 0,$$

$$(n - w) = \pm \sqrt{V} \sqrt{\left(\frac{R_w}{R_n} \right)^2 - 1} = \pm \frac{\sqrt{V}}{R_n} \sqrt{R_w^2 - R_n^2},$$

$$R_w = R_n \sqrt{1 + \frac{(n - w)^2}{V}}, \quad \sqrt{V} = \frac{\pm (n - w) R_n}{\sqrt{R_w^2 - R_n^2}},$$

where $V = \frac{z_1}{z_2}$.

An inspection of the above equations shows that if R_n and R_w are both multiplied by an arbitrary constant, the roots of the equation will be

¹ Assume that Δl_3 corresponds to one turn of W_3 , and denote by n the number of turns from a given position near the top of the screw. Then n and P_1 increase as W_3 is lowered.

unaltered. Therefore the ratios used need only be proportional to the true values of the ratios. Also, if (A) has a constant amplitude, only the readings for θ need to be taken.

From three observations, three corresponding values of n and R_n can be obtained, and the two quantities R_w and w , which will be the same in all cases can be eliminated, thus giving the value of $\sqrt{V}$.

Now,

$$\sqrt{z_1} = \sqrt{V}\sqrt{z_2} = aP_1 - 2a\alpha P,$$

and therefore

$$P_1 = \sqrt{V}\sqrt{\frac{z_2}{a^2}} + 2\alpha P,$$

also

$$P_1 = B \left(\frac{s_1 s_2}{s} \right)^2 + \left\{ B' \left(\frac{s_1 s_2}{s} \right)^2 + k_1 + k_2 \left(\frac{s_1}{s} \right)^2 \right\},$$

$$B = \left(\frac{s}{s_1 s_2} \right)^2 \sqrt{V}\sqrt{\frac{z_2}{a^2}} + 2\alpha P - \left(\frac{s}{s_1 s_2} \right)^2 \left\{ B' \left(\frac{s_1 s_2}{s} \right)^2 + k_1 + k_2 \left(\frac{s_1}{s} \right)^2 \right\},$$

and

$$B = \left(\frac{s}{s_1 s_2} \right)^2 \sqrt{V}\sqrt{\frac{z_2}{a^2}} - K,$$

where K is a positive constant, and depends only on s , s_1 , s_2 , α , P and the friction not due to the sphere. Therefore, if two different spheres of radii r_1 and r_2 are used,

$$B_1 = \left(\frac{s}{s_1 s_2} \right)^2 \sqrt{V_1}\sqrt{\frac{z_2}{a^2}} - K_1,$$

and

$$B_2 = \left(\frac{s}{s_1 s_2} \right)^2 \sqrt{V_2}\sqrt{\frac{z_2}{a^2}} - K_2,$$

where $K_2 - K_1$ is practically zero. Therefore, subtracting the first from the second gives

$$B_2 - B_1 = (\sqrt{V_2} - \sqrt{V_1}) \sqrt{\frac{z_2}{a^2}} \left(\frac{s}{s_1 s_2} \right)^2.$$

From this equation, the difference between B_2 and B_1 can be found, since all the quantities in the second member can be measured. From the theory given by Stokes, this difference has the following value:

$$B_2 - B_1 = 3\pi\sqrt{2a}\sqrt{\mu'} \left\{ (r_2 - r_1) \left(r_2 + r_1 + \sqrt{\frac{a}{2}}\sqrt{\mu'} \right) \right\} \rho,$$

and from this equation μ the coefficient of viscosity can be computed. But the vessel containing the liquid must be large enough, compared to the sphere used so that the assumption on which the equations were

derived are justified, or a correction for the effect of the containing vessel must be made.

Let B_1' and B_2' be the values of B_1 and B_2 corresponding to a vessel of infinite radius. Let λ_1 and λ_2 be coefficients depending on a , μ' , and the ratio of the radius of the sphere to the containing vessel, so that

$$\lambda_1 B_1' = B_1,$$

and

$$\lambda_2 B_2' = B_2,$$

then

$$\lambda_2 B_2' - \lambda_1 B_1' = (\sqrt{V_2} - \sqrt{V_1}) \sqrt{\frac{z_2}{a^2} \left(\frac{s}{s_1 s_2} \right)^2}.$$

Let λ_1'' and λ_2'' correspond to the cases where the radius of the vessel is R'' and R' respectively; the same sphere being used. Then

$$B_1'(\lambda_1'' - \lambda_1') = (\sqrt{V_1''} - \sqrt{V_1'}) \sqrt{\frac{z_2}{a^2} \left(\frac{s}{s_1 s_2} \right)^2}.$$

If this result is zero, then the radius is large enough, and no correction is necessary.

III. EXPERIMENTAL RESULTS.

A gravity pendulum consisting of a wrought iron rod 1.5 cm. in diameter, and 180 cm. in length; and two cylindrical weights, each being of 5 kg. mass, was mounted on a frame supported by two concrete piers. By adjusting the position of the weights, any value of the period, from about 2 to 25 seconds could be secured. By means of a mirror attached to the knife-edge, and a vertical scale and horizontal telescope supported on one of the piers, 115 cm. from the mirror, measurements of the angular displacements were made. The scale was divided into mm., and readings were taken to 0.1 mm. Within the limits of observational errors, the displacement of the pendulum agreed with the following formula:

$$\theta_0 = \left(-\frac{c}{l_1} \right) e^{-at} \sin at.$$

The measuring apparatus, already described, was supported with its knife-edge E parallel to that of the large pendulum,¹ and about 8 cm. above it. A mirror was attached to E above the mirror of the large pendulum, and a second horizontal telescope was mounted above that used for the large pendulum. The same scale was used for both. The apparatus was not enclosed nor shielded from air currents.

A short rigid bar, secured to the large pendulum in a position perpendicular to its knife-edge and pendulum rod, supported at the distance l_1

¹ See Fig. 2 for diagram.

from the knife-edge, by means of a fine wire, the end *A* of the lower beam of the measuring apparatus. Because of this arrangement, the end *A* of the beam *AC* was constrained to have the vertical displacement:

$$y_0 = -l_1\theta_0 = ce^{-at} \sin at.$$

In order to test the capacity of the apparatus for measuring large forces acting on the sphere, dark filtered cylinder oil, grade "N," was used at temperatures from 18° to 22° Cent. At these temperatures, the coefficient of viscosity of the oil is several thousand times as great as that of water.

The numerical values of the constants in the experiment on oil are tabulated below:

Weight of the measuring apparatus, about 400 grams.

Weight in each scale-pan, about 80 grams.

Weight of each scale-pan, about 15 grams.

$$W_3 = 13.52 \text{ gr.}$$

$$M'' = 76.84 \text{ gr.}$$

$$\Delta l_3 = 00.0794 \text{ cm.}$$

$$\sqrt{z_2} = 1,052 \text{ dyne cm.}$$

$$s_1 = 5.0 \text{ cm.}$$

$$s_2 = 2.5 \text{ cm.}$$

$$s_3 = 7.5 \text{ cm.}$$

$$s = 10.0 \text{ cm.}$$

$$a = 1.702 \text{ (sec.)}^{-1}.$$

$$\alpha = 0.0007 \text{ (sec.)}^{-1}.$$

The radius of the cylinder containing the liquid was 5.5 cm.; its depth was 18 cm. The cylinder was practically full of the liquid, and in all cases the top of the sphere was 6.5 cm. below the liquid surface. The maximum velocity of the sphere was less than 0.1 cm./sec., and its maximum displacement was less than 0.07 cm.

The maximum deflections θ' and θ_0' were determined alternately, and the value for θ' was divided by the mean of the two nearest values of θ_0' for a single determination of the ratio R' . The following set of readings and

results is typical of the accuracy of the work. In the first and second columns are the average of the differences of the scale readings for determining θ_0' and θ_0 respectively. The last column contains the average value of R' minus each value.

Diagram is inverted.

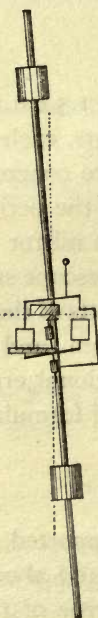


Fig. 2.

		R'	$.5692-R'$			R'	$.5692-R'$
2.590	1.560	.6025	-.0063	2.445	1.460	.5970	-.0008
2.575	1.520	.5910	.0048	2.425	1.460	.6062	-.0058
2.560	1.510	.5900	.0062	2.400	1.430	.5960	.0002
2.540	1.505	.5925	.0037	2.370	1.420	.5980	-.0018
2.515	1.495	.5945	.0017			.5962	
2.480	1.485	.5990	-.0028				

Tabulation of Results.—The temperature of the liquid is denoted by t . Under $\sqrt{V}$ are the results of substituting the given values of n , w , R_n , and R_w in the formula:

$$\sqrt{V} = \frac{\pm (n - w)}{\sqrt{R_w^2 - R_n^2}}.$$

$$t = 18.1^\circ \quad r_1 = .495 \text{ cm.} \quad R_w^2 = 8.30 \quad w = 1.384.$$

n	R_n	R_n^2	$\sqrt{V_1}$	$B_1 + K = 395.7 \sqrt{V_1}$
1	2.612	6.822	2.973	
7	1.5513	2.4065	2.950	
11	.9375	.8789	2.965	
17	.5709	.3259	2.954	
			2.960	1,171

$$t = 18.05^\circ \quad r_2 = .792 \text{ cm.} \quad R_w^2 = 10.12 \quad w = -1.700.$$

n	R_n	R_n^2	$\sqrt{V_2}$	$B_2 + K = 395.7 \sqrt{V_2}$
0	3.123	9.755	8.79	
6	2.392	5.722	8.78	
18	1.296	1.680	8.79	
			8.787	3,477

$$t = 22.3^\circ \quad r_1 = .495 \text{ cm.} \quad R_w^2 = 11.30 \quad w = 2.98.$$

n	R_n	R_n^2	$\sqrt{V_1}$	$B_1 + K$
1	1.9817	3.927	1.445	
7	1.1335	1.2848	1.436	
11	.5792	.33507	1.405	
17	.3423	.11717	1.436	
			1.431	566

$$t = 22.15^\circ \quad r_2 = .792 \text{ cm.} \quad R_w^2 = 9.34 \quad w = 1.50.$$

n	R_n	R_n^2	$\sqrt{V_2}$	$B_2 + K$
0	2.755	7.590	3.125	
6	1.748	3.055	3.137	
12	.9001	.8102	3.237	
18	.5701	.3255	3.136	
			3.159	1,249

$$t = 22.10^\circ \quad r_2 = .792 \text{ cm.} \quad R_w^2 = 8.63 \quad w = 1.73.$$

n	R_n	R_n^2	$\sqrt{V_2}$	$B_2 + K$
1	2.862	8.191	3.155	
7	1.513	2.289	3.142	
11	.9500	.9025	3.160	
17	.5962	.3555	3.162	
			3.155	1,248

The coefficient of viscosity was not computed from the preceding data, because the cylinder was not large enough compared to the spheres for the assumption of a cylinder of infinite radius to be valid for a fluid having such a large coefficient of viscosity. However, the above trials show that the method adopted can be successfully applied, even to the measurement of very large frictional forces.

The apparatus, used in the experiment just described, was not well adapted to the measurement of small forces acting on the sphere, because of its crudeness, and because the friction in the apparatus itself was comparatively large. But in order to test the method through a wide range of conditions, an attempt was made to determine the coefficient of viscosity of water. In this experiment the sphere was suspended close to the point C of the lower bar, in order to increase the effect of the frictional force. Also, since the quantity z_1 was now much smaller than before, the series of values of R was obtained by correspondingly smaller changes in the expression $(P_2 + P[\alpha^2 - a^2] - \alpha P_1)$. The changes were made by adding equal weights to each scale-pan, thus, varying P instead of P_2 as was done before; and by a method similar to that already described the following formula was derived:

$$(n - w)^2 + V \left[1 - \left(\frac{R_w}{R_n} \right)^2 \right] = 0,$$

in which n is the number of multiples of a given weight W_4 added to each scale-pan, $\sqrt{z_2}$ equals the change in the moment of inertia of the system when one of these weights is added to each pan. That is:

$$\sqrt{z_2} = a^2 W_4 (s_1^2 + s_4^2).$$

In this case a decrease in n has the same effect as an increase in the previous case, but this formula can be used in the same way as the previous one, assuming the symbols to have the values indicated above.

The numerical values of the constants in the following experiment are tabulated below:

Weight of the measuring apparatus, about 400 grams.

Weight in each scale-pan, about 50 grams.

Weight of each scale-pan, about 15 grams.

$W_4 = 1.00$ grams. $\sqrt{z_2} = 146.5$ dyne cm.

$s_1 = 5.0$ cm. $s_2 = 8.90$ cm.

$s_3 = 1.1$ cm. $s = 10.0$ cm.

$\rho = 1.000$. $a = 1.711$ (sec.)⁻¹.

The radius of the cylinder containing the liquid was 8 cm.; its depth was 20 cm. The cylinder was practically full of water, and in each case the top of the sphere was 6.5 cm. below the surface of the water. The maximum velocity of the sphere was less than .15 cm./sec., and its maximum displacement was less than .09 cm.

The amplitude of the large pendulum was maintained constant by means of an electro-magnet and the scale readings for determining θ were taken as before. In this case they should be constant. The following set of readings and results is typical of the accuracy of the work. The differences between successive scale readings, denoted by R , are given in the first column. The last column contains the average value of R minus each value.

R	$2.719 - R$	R	$2.719 - R$	R	$2.719 - R$
2.67	+.049	2.74	-.021	2.68	+.039
2.70	+.019	2.73	-.011	2.67	+.049
2.68	+.039	2.68	+.039	2.68	+.039
2.69	+.029	2.69	+.029	2.71	+.009
2.68	+.039	2.72	-.001	2.76	-.041
2.72	-.001	2.73	-.011	2.76	-.041
2.79	-.071	2.69	+.029	2.78	-.061
2.77	-.051	2.72	-.001	2.71	+.009
2.78	-.061	2.71	+.009	2.75	-.031
2.77	-.051	2.71	+.009	2.70	+.019
2.72	-.001	2.70	+.019	Aver. R 2.719	
2.72	-.001				

TABULATION OF RESULTS.

The temperature of the liquid is denoted by t . Under $\sqrt{V}$ are given the results of substituting the values of n , w , R_n , and R_w in the formula:

$$\sqrt{V} = \frac{\pm (n - w)R_n}{\sqrt{R_w^2 - R_n^2}}.$$

$$t = 16.5^\circ \quad r = .495 \text{ cm.} \quad R_w^2 = 85.8 \quad w = 3.474.$$

n	R_n	R_n^2	$\sqrt{V_1}$	$B_1 + K = 2.525 \sqrt{V_1}$
4	6.994	48.95	.6057	
2	3.525	12.42	.6065	
0	1.593	2.538	.6065	
			.606	1.530

$$t = 16.5^\circ \quad r_2 = .952 \text{ cm.} \quad R_w^2 = 42.80 \quad w = 4.34.$$

n	R_n	R_n^2	$\sqrt{V_2}$	$B_2 + K = 2.525 \sqrt{V_2}$
8	1.850	3.422	1.079	
4	6.238	38.92	1.080	
2	2.721	7.405	1.070	
0	1.556	2.420	1.064	
			1.073	2.710

$$B_2 - B_1 = 2.710 - 1.530 = 1.18.$$

The expression $(B_2 - B_1)$ can be transformed into the following:

$$B_2 - B_1 = 3\pi \sqrt{2a} \sqrt{\mu'} \left\{ (r_2 - r_1) \left(r_2 + r_1 + \sqrt{\frac{2}{a}} \sqrt{\mu'} \right) \right\},$$

from which after substituting the values from the above experiment, the following equation for determining μ is obtained:

$$1.18 = 3\pi \sqrt{3.422} \sqrt{\mu'} \{ (.457)(1.447 + 1.08 \sqrt{\mu'}) \}$$

or

$$\mu' + 1.338 \sqrt{\mu'} - .1369 = 0.$$

The solution of this equation gives $\mu = .0091$, while the value .0111 is given by the formula:¹

$$\mu = \frac{.0178}{1 + .0337t + .00021t^2}.$$

Thus there is a fair agreement with other methods, considering the unfavorable working conditions.

IV. SUGGESTIONS FOR FUTURE RESEARCH.²

In the measuring apparatus used in the experiments just described, the force which opposed the displacement of the system but did not

¹ This formula is based on Poiseuille's capillary tube experiment. See Lamb's Hydrodynamics, 3d edition, p. 536.

² Further suggestions for future research are given in Stokes's Math. and Phys. Papers, Vol. III., pp. 123-127.

depend upon the friction, was due partly to the elasticity of the steel strips and partly to gravity. If, instead of steel strips knife-edges were used, the return force would depend upon gravity only and could be made so small, without loss of stability, that a given natural period could be secured by a much lighter apparatus. If, in addition to reducing the weight, agate bearings were provided, the friction of the apparatus would be greatly reduced, and therefore greater accuracy would result. It would also be desirable to protect the apparatus from air currents and dust.

In making measurements, the weight W_3 should be so adjusted that two values of the ratio R are roughly one half the maximum value, one corresponding to the case where $(P_2 + P[\alpha^2 - a^2] - \alpha P_1) = \sqrt{z_2}(n - w)$ is positive and the other when it is negative, and the third value should be near the maximum. The three values of R thus obtained are sufficient for computing the force, but it is desirable to check the work by additional values of R between the first two.

The same apparatus can be used for the measurement of the force acting on any vibrating body, if the force opposes the displacement and is proportional to the velocity, and it is well adapted to the case in which a very small maximum velocity and displacement is required.

The advantages of measuring the coefficient of viscosity by the oscillating sphere method can only be settled by a series of experiments in which the force is measured when the working conditions are in agreement with those required by the theory.

V. SUMMARY OF THE PAPER.

The hydrodynamical theory of a frictionless fluid failed to account for the experimental results of the pendulum observations of Bessel, Baily and Dubuat, and this fact led Stokes to apply the theory when the fluid friction was taken into account.

The predictions based on this theory agreed well with the experiments on pendulums vibrating in gases, but when the fluid was water, instead of gas, the agreement was not very satisfactory. This disagreement was due to the difficulty of making the observations, rather than to the theory.

The fact that the inertia correction could be determined experimentally and that it had been shown how it depended on the coefficient of viscosity led Stokes to suggest that his theory might be employed to calculate the coefficient of viscosity of gases from observations on a pendulum, consisting either of a sphere attached to a fine wire, or of a cylinder.

He also showed how to compute the frictional force acting on a sphere

and a cylinder, vibrating in a viscous fluid. From this result the coefficient of viscosity can be computed from observations which give the retarding force. For this purpose he proposed that the decrement of the arc of oscillation be used. This does not require as accurate a value of the period as the other method.

The greatest difficulty in each method when the coefficient of viscosity is as large as that of water, or larger, is the rapidity with which the oscillations diminish.

The present investigation was undertaken to overcome the difficulties mentioned by Stokes when applying the oscillating sphere method to the measurement of large coefficients of viscosity. This object was accomplished by employing a forced vibration method in which the sphere was suspended by a fine wire, and had a slow motion of translation only along a vertical line, and the oscillations were small, and either diminished very slowly, or were maintained constant for any desired length of time.

The above method was applied to the measurement of the force acting on a sphere in water and in a very viscous oil. In each case a satisfactory measurement of the force was obtained.

It is assumed in the hydrodynamical theory that the velocities and displacements are very small, that the amplitude remains constant, and that there is no rotation of the sphere. In measuring the coefficient of viscosity it is necessary that the apparatus can be easily cleaned, that the required quantities can be accurately measured, and that the conditions can be readily reproduced.

These requirements are fulfilled by the method just described, but the advantages of measuring the coefficient of viscosity by the oscillating sphere method can only be determined by further experiments in which the force is measured when the conditions agree with those required by the theory.

Stanford University,
April, 1911.

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